Non-interacting / 'free' systems

The Hamiltonian of the form:

$$H = \sum_{ij} a^+_i a_j h_{ij}$$

are called non-interacting or free because they correspond to particles that are experiencing an external potential but no inter-particle interaction.

The eigenstates of such Hamiltonian have special properties that are not shared by a more general Hamiltonian that is not quadratic in a / a^+.

Going to a diagonal basis (= eigenbasis)

$$H = \sum_{\alpha} \varepsilon_\alpha \, a^+(u_\alpha) \, a(u_\alpha)$$

Thus an eigenstate is of the form

$$|\psi\rangle = a^+(u_1)\, a^+(u_2) \cdots a^+(u_N)|0\rangle$$

which has eigenenergy $\sum_{\alpha=1}^{N} \varepsilon_\alpha$.

(recall, for bosons, not all u_i have to be distinct).

Let's stick with fermions for now.

Consider the correlation f^n:

$$\langle\psi|\, a^+(x)\, a^+(x')\, a(y)\, a(y')|\psi\rangle$$

with respect to $|\psi\rangle$ above:

Fourier transforming, this is:

$$\sum_{\alpha\beta\gamma\delta} \langle\psi|\, a^+(u_\alpha)\, a^+(u_\beta)\, a(u_\gamma)\, a(u_\delta)|\psi\rangle \, u^*_\alpha(x)\, u^*_\beta(x')\, u_\gamma(y)\, u_\delta(y')$$

So we need to calculate

$$\sum_{\alpha\beta\gamma\delta} \langle u_1 u_2 \cdots u_N|\, a^+(u_\alpha)\, a^+(u_\beta)\, a(u_\gamma)\, a(u_\delta)|u_1 \cdots u_N\rangle \, u^*_\alpha(x)\, u^*_\beta(x')\, u_\gamma(y)\, u_\delta(y')$$

For this to be non-zero, one requires either: $\alpha = \gamma$, $\beta = \delta$ or $\alpha = \delta$, $\beta = \gamma$

$$\Rightarrow \langle \psi | a^+(x)\, a^+(x')\, a(y)\, a(y') | \psi \rangle$$

$$= \sum_{\alpha\beta} \langle u_1 \ldots u_N | a^+(u_\alpha)\, a^+(u_\beta)\, a(u_\alpha)\, a(u_\beta) | u_1 \ldots u_N \rangle\, u^*_\alpha(x)\, u^*_\beta(x')\, u_\alpha(y)\, u_\beta(y')$$

$$+ \sum_{\alpha\beta} \langle u_1 \ldots u_N | a^+(u_\alpha)\, a^+(u_\beta)\, a(u_\beta)\, a(u_\alpha) | u_1 \ldots u_N \rangle\, u^*_\alpha(x)\, u^*_\beta(x')\, u_\beta(y)\, u_\alpha(y')$$

$$= -\left[\sum_{\alpha\neq\beta} u^*_\alpha(x)\, u_\alpha(y)\, u^*_\beta(x')\, u_\beta(y')\right]$$

$$+ \left[\sum_{\alpha\neq\beta} u^*_\alpha(x)\, u_\alpha(y')\, u^*_\beta(x')\, u_\beta(y)\right]$$

$$= -\left[\sum_\alpha u^*_\alpha(x)\, u_\alpha(y)\right]\left[\sum_\beta u^*_\beta(x')\, u_\beta(y')\right]$$

$$+ \left[\sum_\alpha u^*_\alpha(x)\, u_\alpha(y')\right]\left[\sum_\beta u^*_\beta(x')\, u_\beta(y)\right]$$

[note that $\alpha = \beta$ term cancels out]

$$= \langle a^\dagger(x)\, a(y')\rangle \langle a^\dagger(x')\, a(y)\rangle$$

$$- \langle a^\dagger(x)\, a(y)\rangle \langle a^\dagger(x')\, a(y')\rangle$$

$$\langle \psi |\, a^\dagger(x)\, a^\dagger(x')\, a(y)\, a(y')\, |\psi\rangle$$

$$= \langle a^\dagger(x)\, a(y')\rangle \langle a^\dagger(x')\, a(y)\rangle$$

$$- \langle a^\dagger(x)\, a(y)\rangle \langle a^\dagger(x')\, a(y')\rangle$$

This is called 'Wick's theorem' in literature.

Bosons: $|\psi\rangle = |u_1 \ldots u_N\rangle$

$$\langle \psi | \, a^\dagger(x) \, a^\dagger(x') \, a(y) \, a(y') |\psi\rangle$$

As in the fermion case, Fourier transforming,

$$\sum_{\alpha\beta\gamma\delta} \langle u_1 u_2 \ldots u_N | \, a^\dagger(u_\alpha) \, a^\dagger(u_\beta) \, a(u_\gamma) \, a(u_\delta) |u_1 \ldots u_N\rangle$$
$$u^*_\alpha(x) \, u^*_\beta(x') \, u_\gamma(y) \, u_\delta(y')$$

There are three separate cases:

(i) $\alpha = \beta = \gamma = \delta$ (ii) $\alpha \neq \beta$, $\alpha = \gamma$, $\beta = \delta$

(iii) $\alpha \neq \beta$, $\alpha = \delta$, $\beta = \gamma$

$$\Rightarrow \langle \psi | \, a^\dagger(x) \, a^\dagger(x') \, a(y) \, a(y') |\psi\rangle$$

$$= \sum_\alpha \langle u_1 \ldots u_N | \, [a^\dagger(u_\alpha)]^2 \, [a(u_\alpha)]^2 |u_1 \ldots u_N\rangle \; u^*_\alpha(x) \, u^*_\alpha(x') \, u_\alpha(y) \, u_\alpha(y')$$

$$+ \sum_{\alpha \neq \beta} \langle u_1 \ldots u_N | \, a^\dagger(u_\alpha) \, a^\dagger(u_\beta) \, a(u_\alpha) \, a(u_\beta) |u_1 \ldots u_N\rangle \; u^*_\alpha(x) \, u^*_\beta(x') \, u_\alpha(y) \, u_\beta(y')$$

$$+ \sum_{\alpha \neq \beta} \langle u_1 \ldots u_N | \, a^\dagger(u_\alpha) \, a^\dagger(u_\beta) \, a(u_\beta) \, a(u_\alpha) |u_1 \ldots u_N\rangle \; u^*_\alpha(x) \, u^*_\beta(x') \, u_\beta(y) \, u_\alpha(y')$$

Let's calculate these terms one by one:

(i) $\sum_{\alpha} \langle u_1 \cdots u_N | \, [a^+(u_\alpha)]^2 \, [a(u_\alpha)]^2 \, | u_1 \cdots u_N \rangle$
$u^*_\alpha(x) \, u^*_\alpha(x') \, u_\alpha(y) u_\alpha(y')$

$$= \sum_{\alpha} n_\alpha (n_\alpha - 1) \, u^*_\alpha(x) \, u^*_\alpha(x') \, u_\alpha(y) u_\alpha(y')$$

(ii) $\sum_{\alpha \neq \beta} \langle u_1 \cdots u_N | \, a^+(u_\alpha) \, a^+(u_\beta) \, a(u_\alpha) a(u_\beta) \, | u_1 \cdots u_N \rangle$
$u^*_\alpha(x) \, u^*_\beta(x') \, u_\alpha(y) u_\beta(y')$

$$= \sum_{\alpha \neq \beta} n_\alpha n_\beta \, u^*_\alpha(x) \, u^*_\beta(x') \, u_\alpha(y) u_\beta(y')$$

$$= \sum_{\alpha \beta} n_\alpha n_\beta \, u^*_\alpha(x) \, u^*_\beta(x') \, u_\alpha(y) \, u_\beta(y')$$

$$- \sum_{\alpha} n_\alpha^2 \, u^*_\alpha(x) \, u^*_\beta(x') \, u_\alpha(y) \, u_\alpha(y')$$

(iii)
$\sum_{\alpha \neq \beta} \langle u_1 \cdots u_N | \, a^+(u_\alpha) \, a^+(u_\beta) \, a(u_\beta) \, a(u_\alpha) | u_1 \cdots u_N \rangle$
$u^*_\alpha(x) \, u^*_\beta(x') \, u_\beta(y) \, u_\alpha(y')$

$$= \sum_{\alpha \neq \beta} n_\alpha n_\beta \, u^*_\alpha(x) \, u^*_\beta(x') \, u_\beta(y) \, u_\alpha(y')$$

$$= \sum_{\alpha \beta} n_\alpha n_\beta \, u^*_\alpha(x) \, u^*_\beta(x') \, u_\beta(y) \, u_\alpha(y')$$

$$- \sum_{\alpha} n_\alpha^2 \, u^*_\alpha(x) \, u^*_\beta(x') \, u_\alpha(y) \, u_\alpha(y')$$

Collecting the three terms together:

$$\langle \psi | a^+(x) a^+(x') a(y) a(y') | \psi \rangle$$

$$\sum_\alpha n_\alpha (n_\alpha - 1) \, U^*_\alpha(x) U^*_\alpha(x') U_\alpha(y) U_\alpha(y')$$

$$+$$

$$\sum_{\alpha\beta} n_\alpha n_\beta \, U^*_\alpha(x) U^*_\beta(x') U_\alpha(y) U_\beta(y')$$

$$- \sum_\alpha n_\alpha^2 \, U^*_\alpha(x) U^*_\beta(x') U_\alpha(y) U_\alpha(y')$$

$$+$$

$$\sum_{\alpha\beta} n_\alpha n_\beta \, U^*_\alpha(x) U^*_\beta(x') U_\beta(y) U_\alpha(y')$$

$$- \sum_\alpha n_\alpha^2 \, U^*_\alpha(x) U^*_\alpha(x') U_\alpha(y) U_\alpha(y')$$

$$= \underbrace{\left[\sum_\alpha n_\alpha U^*_\alpha(x) U_\alpha(y) \right]}_{\langle a^+(x) a(y) \rangle} \underbrace{\left[\sum_\beta n_\beta U^*_\beta(x') U_\beta(y') \right]}_{\langle a^+(x') a(y') \rangle}$$

$$+ \underbrace{\left[\sum_\alpha n_\alpha U^*_\alpha(x) U_\alpha(y') \right]}_{\langle a^+(x) a(y') \rangle} \underbrace{\left[\sum_\beta n_\beta U^*_\beta(x') U_\beta(y) \right]}_{\langle a^+(x') a(y) \rangle}$$

$$- \sum_\alpha n_\alpha (n_\alpha + 1) U^*_\alpha(x) U^*_\alpha(x') U_\alpha(y) U_\alpha(y')$$

$$= \langle \psi | a^{+}(x) a^{+}(x') \, a(y) \, a(y') | \psi \rangle$$

$$= \langle \psi | a^{+}(x) \, a(y') | \psi \rangle \langle \psi | a^{+}(x') \, a(y) | \psi \rangle$$

$$+ \langle \psi | a^{+}(x) \, a(y) | \psi \rangle \langle \psi | a^{+}(x') \, a(y') | \psi \rangle$$

$$\underbrace{- \sum_{\alpha} (n_\alpha + 1) n_\alpha \, U^*_\alpha(x) \, U^*_\alpha(x') \, U_\alpha(y) \, U_\alpha(y')}_{\text{extra term compared to fermions.}}$$

Therefore the 'Wick's Theorem' does not hold true for eigenstates of free bosons